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Untimely Destruction: Pestilence, War and Accumulation in the Long Run

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Abstract

This paper analyses the effects of disease and warfare on the accumulation of human and physical capital, with special reference to the existence – or otherwise – of poverty traps and steady-state growth paths. We employ an overlapping generation framework in which young adults, confronted with such hazards and motivated by old-age provision and altruism, make decisions about investments in schooling and reproducible capital. We establish that there are stationary constellations of war losses and premature adult mortality such that both backwardness, in the sense there is no investment in human capital through schooling, and steady growth with a fully educated population are possible equilibria. Stronger altruism makes the existence of such a poverty trap less likely. Rather paradoxically, altruism can also rule out a steady-state growth path on which children are fully educated when altruism is expressed only in the form of investment in education, because investment in physical capital then affects only old-age provision, and in balanced growth, all components contributing to welfare must keep the same pace.

Keywords: Premature mortality, capital accumulation and destruction, steady states, poverty traps, overlapping generations

JEL Classification: D91, E13, I15, I25, O11, O41

1 Introduction

Dürer's woodcut, 'The Four Horsemen of the Apocalypse', is a compelling and terrifying vision of the great scourges of humanity from time immemorial. This paper deals with three of them – pestilence, war and death, with their accompanying destruction of human and physical capital. Its particular concern is how these calamities affect the accumulation of capital, with special reference to the existence – or otherwise – of poverty traps and steady-state growth paths. The treatment is necessarily stylised, simple and, in contrast to Dürer's masterpiece, desiccated.

In such a setting, the distinction between human and physical capital is vital. Not only are they complementary in production, but they are also, in general, subject to different, albeit not fully independent, hazard rates. The attendant risks are, moreover, not equally insurable. These considerations weigh heavily in the decision of how much to invest for the future and in what form, with all the ensuing consequences for poverty and material prosperity over the long run.

A few selected examples of such calamities will convey some flavour of the historical dimensions of what is involved. The Black Death carried off about one-third of the entire European population between 1347 and 1352. The so-called 'Spanish influenza' pandemic of 1918-1920 is estimated to have caused at least 50 million deaths globally, mortality among young adults being exceptionally high. In recent times, the AIDS pandemic, far slower in its course like the disease itself, still threatens to rival that figure, despite the improved availability of anti-retroviral therapies. As in Dürer's woodcut, pestilence and war also ride together. Half a million soldiers died in an outbreak of smallpox in the Franco-Prussian War of 1870-71 (Morgan, 2002). For every British soldier who fell in combat in the Crimean War (1854-56), another ten died of dysentery, and in the Boer War (1899-1902), the ratio was still one to five.

War losses in the 20th Century make for especially grim reading. Between 15 and 20 million people died in the First World War, the majority of them young men. Almost two million French soldiers fell, including nearly 30 per cent of the conscript classes of 1912-15. Joining this companionship of death were over 2 million Germans, including almost two of every five boys born between 1892 and 1895 (Keegan, 1999: 6-7), almost a million members of the British Empire's armed forces, and many millions more in those of Imperial Austria, Russia and Turkey. Its continuation, the Second World War, was conducted, in every respect, on a much vaster scale. Most estimates suggest that it resulted in about 50 million deaths, directly and indirectly. Among them were 15 or more million Soviet soldiers and civilians, 6 million Poles (20 per cent of that country's pre-war population) and at least 4 million Germans (Keegan, 1990: 590-1). With these staggering human losses went the razing of German and Japanese cities and massive destruction in the western part of the Soviet Union and the states of Eastern Europe. The catalogue of conflicts in the second half of the 20th Century is also unbearably long, with particularly appalling casualties in South-east Asia and Rwanda.

Great epidemics and wars, of course, capture the headlines and grip the imagination, but the majority of those adults who die prematurely fall victim to low-level, 'everyday' causes, especially in poor countries: notable killers are endemic communicable diseases, accidents, violence and childbirth. These are competing hazards – one dies only once –, but their combined effect is not wholly negligible even in contemporary O.E.C.D. countries. In many poorer ones, it is quite dismaying. According to the WHO (2007), those who had reached the age of 20 in the O.E.C.D. group could expect to live, on average, another 60 years or so, their counterparts in China and India another 50-55 years, and those in sub-Saharan Africa but 30-40. The odds that a 20-year old in the O.E.C.D. group would not live to see his or her 40th birthday were 1 or 2 in a 100, rising to 2.5-5 in a 100 for the 50th birthday. These odds were just a little worse for young Chinese, decidedly worse for young Indians, and for young Africans, much less favourable than those of Russian roulette – in some countries where the AIDS epidemic was raging, indeed, scarcely better than the toss of a fair coin.

Issues and Model

The human and material losses inflicted by these causes, whether they take the form of great epidemics and wars, or endemic communicable diseases and low-level conflict, have long-run as well as immediate economic consequences. We therefore address the following questions.

- How large can these hazards be without calling into existence a poverty trap?
- How large can they be without ruling out the possibility of steady-state growth?

- In what settings are both secular, low-level stagnation and steady-state growth possible equilibria?
- Is balanced growth possible when parents are moved by altruism?
- If so, is stronger altruism conducive to faster steady-state growth?

The overlapping generations (OLG) model offers the most natural framework within which to analyse the long-run consequences of economic behaviour in such environments. In the variant adopted here, there are children, young (working) adults and the old. Young adults decide how much schooling the children will receive and how much to put aside to yield a stock of physical capital in the next period. In doing so, they are bound by certain social norms, which govern the distribution of aggregate current consumption among the three generations constituting the family. Untimely destruction can undo these plans, however carefully laid. The children may die prematurely at some point in young adulthood; and war can wreak havoc on the newly formed capital stock. These losses, if they occur, will reduce the resources available to satisfy claims on consumption in old age in the period that follows. Parents may also be motivated by altruism towards their children, so that premature deaths among them will be felt as a distinct loss quite independently of the ensuing reduction in old-age consumption under the prevailing social norms – and arguably all the more keenly if the children have been well educated. The institutional form within which all this takes place is assumed to be a very large extended family, in which the surviving young adults raise all surviving children. Given such pooling and the level of war losses, the law of large numbers makes the level of consumption in old age virtually certain – for those who survive to enjoy it –, but the idiosyncratic risk of dying earlier remains. War losses are wholly uninsurable and operate much like cohort-specific mortality.

Literature

There is a substantial literature on the relationship between the health of populations and aggregate economic activity. Notable is the general empirical observation that good health has a positive and statistically significant effect on aggregate output (Barro and Sala-I-Martin, 1995; Bloom and Canning, 2000; Bloom, Canning and Sevilla, 2001). What is especially relevant for present purposes, however, is a body of work on the macroeconomic effects of AIDS, in which there are varying points of emphasis. Corrigan, Glomm and Méndez (2004, 2005), for example, adopt a two-generation OLG framework in which the epidemic can affect schooling and the accumulation of physical capital, but expectations about future losses play no role. In two contrasting studies of South Africa, Young (2005) uses a Solovian model to estimate the epidemic's impact on living standards through its effects on schooling and fertility, with a constant savings rate; whereas Bell, Devarajan and Gersbach (2006) apply a two-generation OLG model with pooling through extended families and a central role for expectations, but without a role for physical capital.

Closely related theoretical contributions include Chakraborty (2004), in whose OLG framework endogenous mortality is at centre-stage. Better health promotes growth by improving longevity, and investment in health emerges as a prerequisite for sustained growth. In Boucekkine and Laffargue's (2010) two-period framework with heterogeneous levels of human capital, a rise in mortality among adults in the first period reduces the proportion of young adults with low human capital in the second period because the mortality rate among children at the end of the first rises more sharply in poor families. The number of orphans in the first period increases, however, so that the proportion of young adults with low human capital in the second period will increase if orphans go poorly educated. Bell and Gersbach (2013) analyse growth paths and poverty traps when epidemics take the form of two-period shocks to mortality, paying particular attention to their effects on inequality in nuclear family systems, albeit without a place for physical capital.

A salient feature of these studies is the central importance, if only implicitly, of premature adult mortality. Physical capital, when it does appear, is not subject to the hazards that concern us here. In particular, exponential depreciation at a constant rate in Solovian models does not lend itself to the task of representing the shocks of war losses. To our knowledge, there are no contributions that attempt to analyse the combination of both forms of such premature destruction, which is precisely the object of this paper.

It should be remarked that the paper's theme is broadly related to the existence and relevance of 'balanced growth paths'. The classic problem examined by Uzawa (1961) is whether balanced growth paths exist in neoclassical growth models with capital accumulation, population growth and labour- or capital-augmenting technological progress. As recently shown by Grossman at al. (2016), balanced growth requires either an absence of capital-augmenting technological change or a unitary elasticity of substitution between physical and human capital, in which case the forms of factor-augmenting technical change are all equivalent. In this connection, we explore a complementary balanced growth problem: does balanced growth exist in an OLG framework with endogenous physical capital, human capital accumulation, and altruism? We establish conditions on the utility functions with respect to altruism and consumption that allow balanced growth without imposing strong restrictions on the production technology.

The plan of the paper is as follows. Section 2 lays out the model and specifies the general problem to be solved. Section 3 analyses, in comparative fashion, how changes in mortality rates and war losses influence the economy's path. Section 4 establishes conditions for the existence of a poverty trap, but imposing the condition that steady-state growth is not ruled out as an alternative possibility. With a compatible pair of a stable low-level equilibrium and a steady-state growth path as alternative possible outcomes, conditions for the existence of the latter paths and their character are the subject of Section 5. Section 6 briefly draws together some conclusions.

2 The Model

2.1 The macroeconomic structure

There are three overlapping generations: children, who split their time between schooling and work; young adults, who work full time; and the old, who are active neither economically nor in raising children. The timing of events within each period for a generation t, born in period $t-1$ and becoming young adults in period t, is displayed in Figure.**??**.

All individuals belong to numerous, identical and very large extended families. In each such family, the number of young adults at the beginning of period t is N_t^2. They marry and have children at once. Mortality among children occurs only in infancy, and any child who dies is replaced immediately. After such replacement fertility, each couple within the extended family has $2n_t$ children, all of whom survive into adulthood in the next period. Death then claims both some young adults and some of those who have just entered old age. The surviving young adults rear all children collectively and decide how to allocate the children's time between schooling and work, and the resulting aggregate output between consumption and savings, whereby certain social rules govern the claims of children and the old in relation to the consumption of young adults. The numbers of young adults and their offspring who reach maturity are, therefore,

$$N_t^2 = n_{t-1}N_{t-1}^2 \text{ and } N_t^1 = n_t N_t^2,$$

respectively, where n_t is the net reproduction rate (NNR). The numbers of young and old adults who make claims on output in period t are as follows:

$(1 - q_t^2)N_t^2$ young adults survive to raise all children, and
$(1 - q_t^3)N_t^3$ old adults survive to full old age, where $N_t^3 = (1 - q_{t-1}^2)N_{t-1}^2$

and q_t^a denotes the premature mortality rate among age group ($a = 2, 3$). All adults who do reach full old age in period t die at the end of that period.

The first social rule governing consumption is that each child consumes βc_t^2 ($\beta < 1$) when each surviving young adult consumes c_t^2. The second rule decrees that all surviving old adults obtain together the share ρ of the family's current 'full income', $\bar{Y}_t$, which is the level of output that would result if all children were to work full time. Since the extended family is very large, only the individual risk of premature death remains, so that each surviving old adult will consume

$$c_t^3 = \frac{\rho \bar{Y}_t}{(1 - q_t^3)N_t^3} \tag{1}$$

with certainty.

Output is produced by means of labour (measured in efficiency units) and capital, which is made of the same stuff as output, under constant returns to scale. All individuals are endowed with one unit of time. Each young adult possesses λ_t efficiency units of labour, each child γ units. Each fully educated child ($e_t = 1$) requires $w\,(< 1)$ young adults as teachers, so that the direct cost of providing each child with schooling in the amount $e_t \in [0, 1]$ is $w\lambda_t e_t$, measured in units of human capital. The total endowment of the surviving young adults' human capital is defined to be $\Lambda_t \equiv (1 - q_t^2)N_t^2\lambda_t$; and the amount of labour supplied to the production of the aggregate good is

$$L_t \equiv [(1 - q_t^2 - wn_te_t)\lambda_t + n_t\gamma(1 - e_t)]N_t^2. \tag{2}$$

The aggregate savings of the previous period, S_{t-1}, like the cohort of children entering adulthood, are also subject to losses early in the current one, and what does remain has a lifetime of one period. The capital stock available for current production is therefore $K_t = \sigma_t S_{t-1}$, where $\sigma_t \in (0, 1]$ is the survival rate in period t. The current levels of aggregate output and full income are, respectively,

$$Y_t = F(L_t, \sigma_t S_{t-1}). \tag{3}$$

and, putting $e_t = 0$,

$$\bar{Y}_t \equiv Y(e_t = 0) = F(\Lambda_t + \gamma N_t^1, \sigma_t S_{t-1}),$$

where the function F is assumed to have all the other usual nice properties and

$$\Lambda_t + \gamma N_t^1 \equiv \bar{L}_t$$

is the household's endowment of labour at time t.

Full income is available to finance the consumption of all three generations in keeping with the social rules, savings to provide the capital stock in the next period, and investment in the children's education.

$$P_tc_t^2 + S_t + \rho\bar{Y}_t = Y_t, \tag{4}$$

where $P_t \equiv [(1 - q_t^2) + \beta n_t]N_t^2$ is effectively the price of one unit of a young adult's consumption in terms of output, the numeraire.

The formation of human capital involves the contributions of parents' human capital as well as formal education. The human capital attained by a child on reaching adulthood is assumed to be given by

$$\lambda_{t+1} = z_th(e_t)\lambda_t + 1. \tag{5}$$

The multiplier $z_t(> 0)$ represents the strength with which capacity is transmitted across generations; and it may depend on the number of children each surviving young adult must raise. The function $h(\cdot)$ may be thought of as representing the educational technology, albeit with the fixed pupil-teacher ratio of $1/w$. Let $h(\cdot)$ be an increasing, differentiable function on $[0,1]$, with $h(0) = 0$ and $\lim_{e\to 0+} h'(e) < \infty$. The property $h(0) = 0$ implies that unschooled children attain, as adults, only some basic level of human capital, which has been normalised to unity.

2.2 Preferences and choices

Young adults, who make all allocative decisions, have preferences over lotteries involving current consumption, consumption in old age and, if they are altruistic, the human capital attained by the children in their care. When deciding on an allocation (c_t^2, e_t, S_t), young adults must forecast mortality and destruction rates in the coming period. Armed with such (sharp) forecasts[1] and noting (5), they obtain c_{t+1}^3 from (1), which the law of large numbers renders virtually non-stochastic. The stochastic element in the lotteries in question therefore arises only from the probabilities of not surviving into full old age and, where altruism towards the children is concerned, that they will suffer the misfortune to die prematurely in young adulthood. In this connection, let there be full altruism towards adopted children. The adults' preferences are assumed to be additively separable in $(c_t^2, c_{t+1}^3, \lambda_{t+1})$ and von Neumann-Morgenstern in form:

$$V_t = u(c_t^2) + \delta(1 - q_{t+1}^3)u(c_{t+1}^3) + \frac{b(1 - q_{t+1}^2)}{(1 - q_t^2)} n_t v(\lambda_{t+1}),^{2} \quad (6)$$

where δ is the pure impatience factor and b is a taste parameter for altruism. The utility functions u and v are assumed to be strictly concave. In what follows, it will simplify the exposition by defining

$$\chi_t \equiv \delta(1 - q_{t+1}^3) \text{ and } \nu_t \equiv \frac{b(1 - q_{t+1}^2)n_t}{(1 - q_t^2)} .$$

[1]For a vigorous argument that rational actors must have sharp priors, see Elga (2010).
[2]If only natural children count, the 'adjustment' for adopted children $1/(1 - q_t^2)$ drops out.

The young adults' decision problem is as follows:

$$\max_{(c_t^2, e_t, S_t)} V_t \ \text{ s.t. } (1) - (5),\ c_t^2 \geq 0, e_t \in [0,1], S_t \geq 0. \tag{7}$$

When solving it, they note the current state variables, $(N_t^1, N_t^2, N_t^3, q_t^2, q_t^3, \lambda_t, K_t)$, and the variables to be forecast, $(n_{t+1}, q_{t+1}^2, q_{t+1}^3, \sigma_{t+1})$. Let (c_t^{20}, e_t^0, S_t^0) solve (7), where

$$\begin{aligned} e_t^0 &= e_t^0(\lambda_t, K_t, \mathbf{N}_t, \mathbf{q}_t\,; n_{t+1}, \mathbf{q}_{t+1}, \sigma_{t+1}; \beta, \rho, w, \gamma, \delta, b), \\ S_t^0 &= S_t^0(\lambda_t, K_t, \mathbf{N}_t, \mathbf{q}_t\,; n_{t+1}, \mathbf{q}_{t+1}, \sigma_{t+1}; \beta, \rho, w, \gamma, \delta, b), \end{aligned}$$

and $\mathbf{q}_t = (q_t^2, q_t^3)$.

2.3 Evolution of the economy

The task before us is to analyse the evolution of λ_t and K_t with particular reference to the current environment and its future course. The said evolution is governed by the following pair of difference equations:

$$\begin{aligned} \lambda_{t+1} &= z_t h(e_t^0)\lambda_t + 1 = H(\lambda_t, K_t, \mathbf{N}_t, \mathbf{q}_t\,; n_{t+1}, \mathbf{q}_{t+1}, \sigma_{t+1}; \beta, \rho, w, \gamma, \delta, b) \\ K_{t+1}^0 &= \sigma_{t+1} S_t^0 = G(\lambda_t, K_t, \mathbf{N}_t, \mathbf{q}_t\,; n_{t+1}, \mathbf{q}_{t+1}, \sigma_{t+1}; \beta, \rho, w, \gamma, \delta, b). \end{aligned}$$

The first step is to normalise the system by exploiting the assumption that F is homogeneous of degree one. Let $l_t \equiv L_t/N_t^2$ and $s_t \equiv S_t/N_t^2$, so that (1) and (4) can be written as

$$c_{t+1}^3 = \frac{\rho n_t}{(1 - q_{t+1}^3)(1 - q_t^2)} \cdot F\left[(1 - q_{t+1}^2)\lambda_{t+1}(e_t) + n_{t+1}\gamma, \frac{\sigma_{t+1} s_t}{n_t}\right] \tag{8}$$

and

$$[(1 - q_t^2) + \beta n_t]c_t^2 + s_t + \rho F\left[(1 - q_t^2)\lambda_t + n_t\gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right] = F\left(l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right), \tag{9}$$

respectively. The stock of physical capital available to each surviving young adult in period t is

$$k_t \equiv \frac{\sigma_t s_{t-1}}{(1-q_{t-1}^2)n_{t-1}}. \qquad (10)$$

Normalised output is

$$y_t \equiv F\left((1-q_t^2 - wn_te_t)\lambda_t + n_t\gamma(1-e_t), \frac{\sigma_t s_{t-1}}{n_{t-1}}\right).$$

It will be likewise useful to have an analogous definition of normalised full income:

$$\bar{y}_t \equiv F\left(\bar{l}_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right),$$

where $\bar{l}_t \equiv \bar{L}_t/N_t^2$ denotes the normalised endowment of labour at time t. In what follows, it should be noted that F is homogeneous of degree one, in particular, in the quantities $l_t = (1-q_t^2 - wn_te_t)\lambda_t + n_t\gamma(1-e_t)$ and $(1-q_{t-1}^2)k_t$.

Together with the constraints $c_t^2 \geq 0, e_t \in [0,1]$ and $s_t \geq 0$, the budget identity (9) defines the set of all feasible allocations (c_t^2, e_t, s_t). Upon substitution for c_{t+1}^3 from (8) into (6), it is seen that V_t is likewise defined in the same space.

3 The Impact of Mortality and Destruction

We start the analysis by establishing how changes in mortality and destruction rates affect the feasible set and the preference map. Beginning with the former, inspection of (9) reveals that a higher destruction rate of savings (a lower value of σ_t) will cause the feasible set to contract if ρ is not too close to one and, for the subset all allocations in which there is not a heavy resort to child labour (e_t sufficiently close to one), human and physical capital are fairly good substitutes in production (equivalently, the cross-derivative F_{12} is sufficiently small, where F_i denotes the derivative w.r.t. argument i).

An increase in the mortality rate q_t^2 has slightly more complicated effects. For small changes therein, the third term on the l.h.s. of (9) decreases by an amount proportional to $\rho F_1\left(\bar{l}_t, \sigma_t s_{t-1}/n_{t-1}\right)\lambda_t$ and the r.h.s. by an amount $F_1(e_t)\lambda_t$, which is absolutely larger for all $e_t \in [0,1]$. The normalised relative price $[(1-q_t^2)+\beta n_t]$

changes one for one with q_t^2, but with opposite sign. Thus, a higher mortality rate among young adults not only results in a contraction of the feasible set, but also makes current consumption cheaper relative to investment in human and physical capital, whereby it favours the latter: the associated marginal rate of transformation is

$$MRT_{se} = \frac{-1}{n_t(w\lambda_t + \gamma)F_1[(1 - q_t^2 - wn_te_t)\lambda_t + n_t\gamma(1 - e_t),\, \sigma_t s_{t-1}/n_{t-1}]},$$

from which it is seen that an increase in q_t^2 cheapens s_t relative to e_t.

Turning to preferences, the forecast values of all mortality and destruction rates affect V_t through c_{t+1}^3; the current and forecast levels of q^2 enter into the effective weight on the altruism term. By inspection of (8), a higher forecast destruction rate lowers the weight on s_t, and hence on the provision for c_{t+1}^3, but it also makes e_t more attractive than s_t in such provision. An increase in the forecast level of premature mortality among young adults has the opposite effect on both; for if there are fewer survivors, each will produce, *cet. par.*, more full income, and so increase c_{t+1}^3 under the social norm. Indeed, equiproportional changes of the same sign in the survival rates $1 - q_{t+1}^2$ and σ_{t+1} have virtually no effect on c_{t+1}^3 for sufficiently large values of λ_t, as would hold on a steady-state growth path.

Current mortality at the beginning of old age, q_t^3, has no effect on the current feasible set, but its forecast level plays a central role in influencing the weights in V_t. Since u is strictly concave, $c_{t+1}^3 = (1 - q_{t+1}^3)u(\rho n_t F(\bar{\cdot}_{t+1})/(1 - q_t^2)(1 - q_{t+1}^3))$ is decreasing in q_{t+1}^3 for any given (e_t, s_t): if the chances of surviving into old age fall, providing for old age becomes less attractive, while leaving the relative attractiveness of e_t and s_t as vehicles for such provision unchanged. There remains the altruism term in (6), whose effective weight depends on the ratio of forecast to current survival rates among young adults. In a stationary environment, this will be constant.

4 Stationary Equilibria and Poverty Traps

Stationary equilibria can hold only in a stationary environment. Fertility and mortality rates are necessarily constant; but the population need not be stationary

– equivalently, the net reproduction rate (NNR) n_t need not be unity. Given these demographic conditions, output per head can increase only if there is some form of technical progress. If time t does not appear as an explicit argument of F, the only possible form of technical progress in the present framework is the labour-augmenting kind, which is expressed by an increase in the average level of human capital possessed by those supplying labour to production. The first question to be answered, therefore, is whether allocations in which no generation receives any schooling can be equilibria, with the result that $\lambda_t = 1\ \forall t$. An outcome in which young adults are wholly uneducated will be termed a state of backwardness. The second, related question is whether such a state is locally stable. If it is, then backwardness – should it once occur – will persist: that is, there is a poverty trap.

The third question is whether, in any stationary setting, there exist other equilibria, in which children receive at least some schooling. Of particular interest is the happy outcome in which they are fully educated ($e_t = 1$), and once achieved, this outcome continues from then onwards. Now, unbounded growth of output per worker is possible only if λ_t can grow without bound. Recalling (5), it is seen that a necessary condition for this outcome when full education for all is achieved in some period t' and maintained thereafter is $z_t h(1) > 1\ \forall t \geq t'$. In a stationary environment, z_t will be constant, at z, and unbounded growth of output per worker will be impossible if $zh(1) \leq 1$. The fourth and final question, which is of central importance, is whether, in a given stationary setting, both the extremes of backwardness and steady-state growth of output per head can be outcomes in equilibrium. This section will deal with the first two questions, albeit with an eye on the condition $zh(1) > 1$, so as to leave the door open to the third and fourth questions, which will be taken up in Section 5.

4.1 Conditions for poverty traps

We examine the choice of young adults in period t regarding e_t when they expect the next generation to choose $e_{t+1} = 0$. If and only if their optimal choice is $e_t = 0$ will $\lambda_t = 1\ \forall t$ be a steady state of the economy. It will be helpful to rewrite V_t as a function of the decision variables:

$$V_t = u(c_t^2) + \chi_t u\left(\frac{\rho n_t \bar{y}_{t+1}}{(1-q_{t+1}^3)(1-q_t^2)}\right) + \nu_t v(z_t h(e_t)\lambda_t + 1). \tag{11}$$

The associated Lagrangian is

$$\Phi_t = V_t + \mu_t[y_t - [(1-q_t^2) + \beta n_t]c_t^2 - s_t - \rho\bar{y}_t]. \tag{12}$$

The usual assumptions on u ensure that, at the optimum, $c_t^2 > 0$. By assumption, physical capital is necessary in production. Hence, if some young adults are forecast to survive into full old age ($q_{t+1}^3 < 1$), so that $\chi_t > 0$, then $s_t^0 > 0$. Re-expressing the budget constraint in normalised form, as given by (9), the associated f.o.c. are

$$\frac{\partial \Phi_t}{\partial c_t^2} = u'(c_t^2) - \mu_t[(1-q_t^2) + \beta n_t] = 0, \tag{13}$$

$$\frac{\partial \Phi_t}{\partial e_t} = \frac{\delta u'(c_{t+1}^3)\cdot n_t\rho}{(1-q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial \lambda_{t+1}}\frac{\partial \lambda_{t+1}}{\partial e_t} + \nu_t v'(\lambda_{t+1})\frac{\partial \lambda_{t+1}}{\partial e_t} + \mu_t\frac{\partial y_t}{\partial e_t} \leq 0,\ e_t \geq 0, \tag{14}$$

$$\frac{\partial \Phi_t}{\partial s_t} = \frac{\delta u'(c_{t+1}^3)\cdot n_t\rho}{(1-q_t^2)} \cdot \frac{\partial \bar{y}_{t+1}}{\partial s_t} - \mu_t = 0, \tag{15}$$

where

$$\frac{\partial \lambda_{t+1}}{\partial e_t} = z_t h'(e_t)\lambda_t\,,\ \frac{\partial \bar{y}_{t+1}}{\partial s_t} = \frac{\sigma_{t+1}}{n_t}\cdot F_2\left[\bar{l}_{t+1}, \frac{\sigma_{t+1}s_t}{n_t}\right],\ \text{and}$$
$$\frac{\partial y_t}{\partial e_t} = -(\gamma + w\lambda_t)n_t \cdot F_1\left[l_t, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right].$$

We seek to establish conditions that yield a steady-state path $e_t^0 = 0\ \forall t$. Along such a path,

$$\bar{y}_t = y_t(e_t^0 = 0) = F\left[(1-q_t^2) + n_t\gamma, \frac{\sigma_t s_{t-1}}{n_{t-1}}\right]\ \forall t,$$

since $\lambda_t = z_t h(0)\lambda_{t-1} + 1 = 1\ \forall t$. By definition, n_t and the destruction and mortality rates are constant. The index t may now be dropped without ambiguity. Substituting for μ_t from (13) into (15), we have

$$\frac{\delta\rho\sigma[(1-q^2)+\beta n]}{(1-q^2)}F_2\left[(1-q^2) + n\gamma, \frac{\sigma s}{n}\right]u'(c^3) - u'(c^2) = 0. \tag{16}$$

We turn to the budget constraint, noting that when $e_t = 0$, (9) specialises to

$$[(1-q^2) + \beta n]c^2 + s = (1-\rho)F[(1-q^2) + \gamma n, \sigma s/n], \tag{17}$$

and (8) to

$$c^3 = \frac{n\rho}{(1-q^2)(1-q^3)} \cdot F[(1-q^2)+\gamma n, \sigma s/n]. \tag{18}$$

Remark: $F\left[(1-q^2)+\gamma n, \sigma s/n\right]$ is the output per young adult appearing at the *start* of each period. Each of them has n children, but only the fraction $(1-q^2)$ of these adults survive early adulthood. The deceased make no claims on full income in the following period.

Substituting for c^2 and c^3 from (17) and (18), respectively, in (16), we obtain an equation in s, given the constellation (n, q^2, q^3, σ) and the parameters $(\rho, \beta, \gamma, \delta)$. Denote the smallest positive value of s by $s^b = s^b(n, q^2, q^3, \sigma)$.

The final step is to examine (14), with $e_t = 0 \ \forall t$. Substituting for μ_t from (13) in (14) and rearranging terms, we have

$$\left[\rho\,\delta F_1\big(\bar{l}, \frac{\sigma s}{n}\big)u'(c^3) + bv'(1)\right] zh'(0) - \frac{(\gamma+w)F_1\big(\bar{l}, \frac{\sigma s}{n}\big)u'(c^2)}{(1-q^2)+\beta n} \leq 0.$$

Rearranging and using (16), this may be written as

$$\left[\frac{1-q^2}{\sigma} \cdot \frac{F_1\big(\bar{l}, \frac{\sigma s}{n}\big)}{F_2\big(\bar{l}, \frac{\sigma s}{n}\big)}u'(c^2) + \big((1-q^2)+\beta n\big)\, bv'(1)\right] zh'(0) - (\gamma+w)F_1\big(\bar{l}, \frac{\sigma s}{n}\big)u'(c^2) \leq 0,$$

or

$$\left((\gamma+w) - \frac{1-q^2}{\sigma} \cdot \frac{zh'(0)}{F_2\big(\bar{l}, \frac{\sigma s}{n}\big)}\right) u'(c^2)F_1\big(\bar{l}, \frac{\sigma s}{n}\big) \geq \big[(1-q^2)+\beta n\big]\, bv'(1)zh'(0), \tag{19}$$

where F and its derivatives are evaluated at the arguments $((1-q^2)+\gamma n, \sigma s^b/n)$.

A necessary condition that (19) hold as a strict inequality at the hypothesised $e_t^0 = 0$ is

$$\sigma F_2\left[(1-q^2)+\gamma n, \sigma s^b/n\right] > \frac{1-q^2}{\gamma+w} \cdot zh'(0). \tag{20}$$

The sum of the opportunity and direct costs of education at the margin, measured in units of human capital, when $\lambda = 1$ is $(\gamma + w)$ for each child, which is certainly less than unity. A small investment in a child's education will yield $zh'(0)$ units of human capital, over and above the basic endowment of unity, in the next period, with the fraction $1 - q^2$ of all children surviving early adulthood. If h is concave, $zh'(0) \geq zh(1)$, with equality only if h is proportional to e_t (by assumption, $h(0) = 0$): that is, $h'(0)$ is at least one. Since v is strictly concave, however, h may be weakly convex without violating the requirement that V_t be concave over the whole feasible set. This is particularly relevant for sufficiently small values of e_t; for reflection on the educational process and how children learn suggests that $h'(0)$ is indeed modest in size, with admissible values less than one. In this connection, recall that $zh(1) > 1$ is a necessary condition for unbounded growth in output per head to be possible.

The marginal product of physical capital is a pure number, since capital is made of the same stuff as output. When adjusted by the survival rate σ, it measures the yield of investing a little more, instead, in physical capital, the proportional claim on future full income being ρ for both forms of investment. Hence, σF_2 is the opportunity cost of investing a little in education, considering only making provision for one's old age. Now, for any input bundle $((1 - q^2) + \gamma n, \sigma s^b/n)$, F_2 will be large if the production technology is efficient in the sense of exhibiting a high level of total factor productivity. Such a property is quite separate from an efficient educational technology, in the sense that $zh(1) > 1$. It follows that condition (20) can be satisfied if F is sufficiently efficient and both h' and $|h''|$ are sufficiently small. By inspection, if (20) does hold, then it will do likewise for all values of λ_t sufficiently close to 1.

In the absence of altruism ($b = 0$), condition (20) is also sufficient to ensure the existence of a locally stable, steady-state equilibrium in which there is no investment in human capital, children work full time, and output per head is stationary. It does not, however, rule out $zh(1) > 1$, and hence the possible existence of a steady-state path along which output per head grows without limit. If condition (20) holds strongly, then by continuity, the same conclusions will also hold if the altruism motive is weak, since the latter implies that the r.h.s of (19) will be small. If, however, altruism is strong, such a low-level equilibrium may well not exist.

Conclusion: Conditions (19) and $zh(1) > 1$ are compatible, especially if altruism is not too strong and the survival rates for investments in both forms of capital are similar. If the former condition holds as a strict inequality, there will be a poverty trap. If both hold, escape from the trap can be followed by an asymptotic approach to a steady-state growth path along which output per head increases without bound.

4.2 Functional conditions allowing growth as an alternative

We seek to establish more precise conditions for the existence of poverty traps when $zh(1) > 1$, so that unbounded growth is also, in principle, possible. This naturally involves stronger assumptions. The following conditions must be satisfied.

(i) $Z(e_t) \equiv v[zh(e_t)\lambda_t + 1]$ is concave $\forall e_t \in [0, 1]$. This ensures that V_t is concave over the feasible set.

(ii) Condition (20) holds, so that 'backwardness' can be an equilibrium.

(iii) $zh(1) > 1$, to allow unbounded growth when $e_t^0 = 1\ \forall t$ is optimal.

We next explore those conditions in detail and examine whether they can be met simultaneously.

Condition (i). The first and second derivatives are, respectively,

$$
\begin{aligned}
Z' &= v' \cdot zh' \cdot \lambda_t, \\
Z'' &= v'' \cdot (z\lambda_t \cdot h')^2 + v' \cdot z\lambda_t \cdot h'' = v'z\lambda_t \left[\frac{v''}{v'} \cdot z\lambda_t \cdot h'^2 + h'' \right], \\
&= v'z\lambda_t \left[\frac{v''}{v'} \cdot (\lambda_{t+1} - 1)\frac{h'^2}{h} + h'' \right].
\end{aligned}
$$

We next assume functional forms.
A1. Let $v(\lambda_{t+1})$ be iso-elastic, in the form $v \equiv (\lambda_{t+1} - 1)^{1-\eta}/(1-\eta), \quad \eta > 0$.
Then,

$$Z'' = v'z\lambda_t \left[-\eta \cdot \frac{h'^2}{h} + h'' \right] \equiv H(e) \cdot z\lambda_t v',$$

so that $sgn\ Z'' = sgn\ H(e_t)$.

A2. Let $h(e_t) = a_1 e_t + a_2 e_t^2/2 - a_3 e_t^3/3, \quad (a_1, a_2, a_3) \gg \mathbf{0}$.
Then,

$$h' = a_1 + a_2 e_t - a_3 e_t^2,\ h'' = a_2 - 2a_3 e_t.$$

It is seen that h' is increasing on $[0, \min\{\frac{a_2}{2a_3}, 1\})$ and decreasing on $(\min\{\frac{a_2}{2a_3}, 1\}, 1]$. Equivalently, $h'' \gtreqless 0$ according as $e_t \gtreqless a_2/2a_3$. If, further, $h'(0) = a_1 > 0$ and $h'(1) = a_1 + a_2 - a_3 > 0$, we must have $h'(e_t) > 0 \quad \forall e_t \in [0,1]$: h is strictly increasing on the whole interval.
Remark: By allowing $a_2 > 0$, we introduce a strictly convex section of $h(e_t)$ over the interval $[0, a_2/2a_3)$.

Under *A2*, we have

$$H(e_t) = -\eta \cdot \frac{(a_1 + a_2 e_t - a_3 e_t^2)^2}{(a_1 + a_2 e_t/2 - a_3 e_t^2/3)e_t} + (a_2 - 2a_3 e_t).$$

Since $a_1 > 0$, $H(e_t) < 0$ for all e_t sufficiently close to zero. Indeed, $|H(e_t)|$ becomes arbitrarily large as $e_t \to 0$, and since $(a_2 - 2a_3 e_t)$ declines linearly with e_t, there exists a measurable set $S^h = \{\mathbf{a}, \eta) : \mathbf{a} \gg \mathbf{0},\ a_1 + a_2 > a_3,\ \eta > 0\}$ s.t. $H(e_t) < 0 \quad \forall e_t \in [0,1]$.

Condition (ii). In condition (20), s^b is chosen at $e_t = 0$, but the exact form of $h(e_t)$ has no effect on s^b provided that form is also compatible with a growth path along which $e_t^0 = 1\, \forall t$. It follows that (20) will be satisfied if a_1 is sufficiently close to zero.

Condition (iii). It remains to be demonstrated that there are members of the set S^h satisfying not only (20), but also $zh(1) > 1$, that is,

$$z(a_1 + a_2/2 - a_3/3) > 1. \tag{21}$$

Using this inequality in the r.h.s. of (20), we have

$$\frac{1-q^2}{\gamma + w} \cdot z a_1 > \frac{1-q^2}{\gamma + w} \cdot \frac{a_1}{a_1 + a_2/2 - a_3/3}.$$

Choose $\mathbf{a}$ s.t. (21) just holds, i.e., the growth rate $g \equiv zh(1) - 1$ is barely positive, so that the r.h.s. of (20) is arbitrarily close to

$$\frac{1-q^2}{\gamma + w} \cdot \frac{a_1}{a_1 + a_2/2 - a_3/3}.$$

As a final step, consider the following members of the family defined by *A2*:

$$a_1 = 0.1,\ a_2 = a_3 = 1;\ \frac{a_1}{a_1 + a_2/2 - a_3/3} = \frac{3}{8},$$
$$a_1 = 0.1,\ a_2 = 2,\ a_3 = 1;\ \frac{a_1}{a_1 + a_2/2 - a_3/3} = \frac{3}{23}.$$

In the latter case, the r.h.s. of (20) is barely larger than $((1 - q^2)/(\gamma + w))\frac{3}{23}$, which is surely smaller than the l.h.s. of (20) if $F(\cdot)$ is fairly productive and the destruction rate $1 - \sigma$ is sufficiently low.

Conclusion. With this final step we have established that if the sub-utility function v is iso-elastic, F is sufficiently productive and $1 - \sigma$ is sufficiently small, then there exists a measurable subset of the family of functions satisfying *A2* such that conditions (i), (ii) and (iii) are satisfied.

5 Steady-state Growth Paths

We now turn to the opposite extreme, in which all children have been fully educated for countless generations and the human capital of a young adult is growing steadily at the intergenerational rate $g = zh(1) - 1 > 0$, as are c_t^2, c_t^3 and s_t. Correspondingly, inputs of human and physical capital and aggregate output are all growing at the rate $n(zh(1) - 1)$. Under what conditions, if any, does such a hypothetical growth path exist?

Suppose the economy is on such a path. The pairwise marginal rates of transformation among c_t^2, e_t and s_t are obtained from the budget constraint (9). For any value of $e_t \in [0, 1]$,

$$MRT_{ce} = -\frac{(1 - q^2) + \beta n}{n(w\lambda_t + \gamma)F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]}, \tag{22}$$

$$MRT_{se} = -\frac{1}{n(w\lambda_t + \gamma)F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]}, \tag{23}$$

$$MRT_{cs} = -[(1 - q^2) + \beta n], \tag{24}$$

where $F_1\left[l_t, \frac{\sigma s_{t-1}}{n}\right]$ will be constant along the hypothesised path.

Turning to preferences, total differentiation of (11) yields

$$dV_t = u'(c_t^2)\cdot dc_t^2 + n\Big(\delta\rho F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + bv'(\lambda_{t+1})zh'(e_t)\lambda_t\Big)\cdot de_t$$
$$+ \frac{\delta\rho\sigma}{1-q^2}F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)\cdot ds_t\,.$$

The corresponding marginal rates of substitution are

$$MRS_{ce} = -\frac{u'(c_t^2)}{n\big(\delta\rho F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big)zh'(e_t)\lambda_t} \equiv -\frac{u'(c_t^2)}{Q_t zh'(e_t)\lambda_t} \equiv -R_t\,, \tag{25}$$

$$MRS_{se} = -\frac{\delta\rho\sigma F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)}{(1-q^2)Q_t zh'(e_t)\lambda_t}\,, \tag{26}$$

$$MRS_{cs} = -\frac{(1-q^2)u'(c_t^2)}{\delta\rho\sigma F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)}\,, \tag{27}$$

where the marginal products $F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)$ and $F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big))$ are constant along the hypothesised path. The ratio c_t^3/c_t^2 will be constant, whose value is denoted by κ.

As in Section 4.2, we assume a specific functional form, now for the sub-utility function u.
A3. Let u be iso-elastic: $u = c^{1-\xi}/(1-\xi)$.
Then,

$$u' = c^{-\xi}, u'' = -\xi c^{-(\xi+1)}, \text{ and } u''c = -\xi c^{-\xi} = -\xi u'.$$

Hence, along the hypothesised path, $u'(c_t^2)/u'(c_{t+1}^3) = [\kappa(1+g)]^{\xi}$. Since $c_t^2 > 0$ and $s_t > 0$, it follows from (24) and (27) that

$$(1-q^2) + \beta n = \frac{(1-q^2)[\kappa(1+g)]^{\xi}}{\delta\rho\sigma F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)}. \tag{28}$$

Given that steady-state growth has been long established, λ_t is so much larger than γ that $F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)$ depends only on the ratio $(1-q^2)\lambda_{t+1}/(\sigma s_t/n) = \lambda_{t+1}/k_{t+1}$, which is a constant along the path in question. Hence, given $(n, q^2, q^3, \sigma; \beta, \delta, \rho)$ and the technologies zh and F, (28) yields the (unique) steady-state value of λ_t/k_t. This is a central result.

We now examine the marginal condition involving (c_t^2, e_t). That is to say, we compare the levels of MRT_{ce} and MRS_{ce} along the path in question, noting that it involves $e_t = 1\,\forall t$. Differentiating (9) and (25) totally and noting that

$$ds_t/s_t = dk_t/k_t = d\lambda_t/\lambda_t = dc_t^2/c_t^2 \tag{29}$$

along this path, we obtain, after some manipulation,

$$\frac{dR_t}{R_t} = \left[\frac{u''(c_t^2)c_t^2}{u'(c_t^2)} \cdot \frac{\left[(1-q^2-wn)F_1\big(l_t, \frac{\sigma s_{t-1}}{n}\big) - \rho F_1\big(\bar{l}_t, \frac{\sigma s_{t-1}}{n}\big)\right]\lambda_t - s_t}{[(1-q^2)+\beta n]c_t^2} - \frac{dQ_t}{d\lambda_t}\frac{\lambda_t}{Q_t} - 1\right]\frac{d\lambda_t}{\lambda_t}\,. \tag{30}$$

Total differentiation of $Q_t = n\big(\delta\rho F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big)$ yields, noting (29) once more,

$$\begin{aligned} dQ_t = {} & n\left[\delta\rho\left((1-q^2)F_{11}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + \frac{u''(c_{t+1}^3)c_{t+1}^3}{zh(1)\lambda_t}F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)\right) + bv''(\lambda_{t+1})\right] \\ & \cdot zh(1)\, d\lambda_t + \delta\rho\sigma F_{12}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)ds_t \\ \equiv {} & A' \cdot zh(1)\cdot d\lambda_t + \delta\rho\sigma F_{12}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)ds_t\,. \end{aligned} \tag{31}$$

Recalling *A1*, we have

$$v''(\lambda_{t+1}) = -\eta\frac{v'(\lambda_{t+1})}{\lambda_{t+1}-1} = -\eta\frac{v'(\lambda_{t+1})}{zh(1)\lambda_t}\,. \tag{32}$$

We examine next the expression $\dfrac{dQ_t}{d\lambda_t}\cdot\dfrac{\lambda_t}{Q_t}$ on the r.h.s. of (30). From (31), we have

$$\frac{dQ_t}{d\lambda_t}\cdot\frac{\lambda_t}{Q_t} = \frac{A'zh(1)\cdot\lambda_t + \delta\rho\sigma F_{12}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)s_t}{n\big(\delta\rho F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)u'(c_{t+1}^3) + bv'(\lambda_{t+1})\big)}\,.$$

Collecting terms in the numerator involving $u'(c_{t+1}^3)$ and recalling *A3*, we obtain

$$J \equiv n\delta\rho\left[(1-q^2)\lambda_{t+1}F_{11}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big) - \xi F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big) + \frac{\sigma s_t}{n}F_{12}\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)\right]\,.$$

Since $F_1\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)$ and $F_2\big(\bar{l}_{t+1}, \frac{\sigma s_t}{n}\big)$ are homogeneous of degree zero, it follows from

Euler's Theorem that

$$[(1-q^2)\lambda_{t+1}+n\gamma]F_{11}\big(\bar{l}_{t+1},\frac{\sigma s_t}{n}\big)+\frac{\sigma s_t}{n}F_{12}\big(\bar{l}_{t+1},\frac{\sigma s_t}{n}\big)=0,$$

so that for sufficiently large λ_t, J reduces to $-n\xi\delta\rho F_1\big(\bar{l}_{t+1},\frac{\sigma s_t}{n}\big)$. Hence, substituting for $v''(\lambda_{t+1})$ from (32), we obtain the elasticity of Q_t w.r.t. λ_t:

$$\frac{dQ_t}{d\lambda_t}\frac{\lambda_t}{Q_t}=\frac{-\xi\delta\rho F_1\big(\bar{l}_{t+1},\frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)-\eta b v'(\lambda_{t+1})}{\delta\rho F_1\big(\bar{l}_{t+1},\frac{\sigma s_t}{n}\big)u'(c_{t+1}^3)+bv'(\lambda_{t+1})}\,.$$

By hypothesis, (λ_t, s_t) are growing at the rate $g=zh(1)-1$. Hence, this elasticity can be expressed in the form

$$\frac{dQ_t}{d\lambda_t}\cdot\frac{\lambda_t}{Q_t}=-\frac{\xi A+\eta B(1+g)^{-(\eta-\xi)}}{A+B(1+g)^{-(\eta-\xi)}}\,, \tag{33}$$

where A and B are positive constants.

There remains the expression multiplying $u''(c_t^2)\cdot c_t^2/u'(c_t^2)\,(=-\xi)$ on the right-hand side of (30), namely,

$$M\equiv\frac{\big[(1-q^2-wn)F_1\big(l_t,\frac{\sigma s_{t-1}}{n}\big)-\rho F_1\big(\bar{l}_t,\frac{\sigma s_{t-1}}{n}\big)\big]\lambda_t-s_t}{[(1-q^2)+\beta n]c_t^2}\,, \tag{34}$$

which is a constant in virtue of the hypothesis that c_t, s_t and λ_t are growing at the same rate. Recalling (9) and that F is homogeneous of degree one, a little manipulation reveals that $M \gtrless 1$ according as

$$\rho F_2\big(\bar{l}_t,\frac{\sigma s_{t-1}}{n}\big)=\rho F_2((1-q^2)\lambda_t+\gamma n,(1-q^2)k_t)\gtrless F_2[(1-q^2-wn)\lambda_t,(1-q^2)k_t)].$$

In practice, wn is very unlikely to exceed 0.1, and on such a growth path γn will be negligible in comparison with λ_t. The fractional claim ρ on full income is also rather unlikely to exceed one-third. It follows that, unless human and physical capital are extremely poor substitutes in production and their steady state ratio k_t/λ_t is very small, so that F_{12} would be correspondingly very large, M is almost surely less than one, albeit plausibly rather close to one. For the ratio

$$(1-q^2)\left(F_2\big(l_t,\frac{\sigma s_{t-1}}{n}\big)-\rho F_2\big(\bar{l}_t,\frac{\sigma s_{t-1}}{n}\big)\right)k_t/[(1-q^2)+\beta n]c_t$$

involves a *difference* in capital's share as the numerator, but the combined consumption of a young adult and children as denominator, so that their ratio is surely rather small.

The final step is to establish conditions under which the choice $e_t = 1$, once attained, remains optimal as λ_t grows without bound at the rate g. It is seen from (22) that when $e_t = 1$ thus holds, $|MRT_{ce}|$ goes to zero at the rate g, so that $\lim_{\lambda_t \to \infty} \left[\dfrac{d(\log |MRT_{ce}|)}{d(\log \lambda_t)} \right] = -1$. To maintain the optimality of $e_t = 1$, however, the $|MRS_{ce}|$ $(= R_t)$ must fall at least as fast as the $|MRT_{ce}|$ as λ_t grows. We now rewrite (30) as

$$\frac{dR_t}{d\lambda_t}\frac{\lambda_t}{R_t} = -\xi M + \frac{\xi A + \eta B(1+g)^{-(\eta-\xi)}}{A + B(1+g)^{-(\eta-\xi)}} - 1.$$

It follows that the required condition is

$$\xi M \geq \frac{\xi A + \eta B(1+g)^{-(\eta-\xi)}}{A + B(1+g)^{-(\eta-\xi)}} \,. \tag{35}$$

If, as is almost surely the case in practice, $M < 1$, it is seen that this condition will be violated if $\xi = \eta$.

Examination of condition (35) reveals that the said condition requires that $\xi M \geq \eta$ if $M < 1$, though sufficiency is also ensured only when $(2M - 1)\xi \geq \eta$. If M is close to 1, then ξ may be close to η, even though ξ must be greater than η. What is the intuition for this result? If there is only human capital, it can be shown that the required condition is $\xi \geq \eta$, with equality as the limiting case. Given the option of providing for old age though saving and the social rule expressed by the parameter ρ, educating the children becomes less pressing in this regard, so that altruism has to work that much harder to maintain $e_t = 1$; for λ_{t+1} is an argument of $v(\cdot)$, but k_{t+1} $(= \sigma s_t / n(1 - q^2))$ is not. It follows that v must be less strongly concave than u if steady-state growth with a fully educated population is to be possible. If, however, parents are perfectly selfish ($\eta = 0$), condition (35) specialises to

$$M \geq \frac{A}{A + B(1+g)^{\xi}} \,,$$

which is more easily satisfied, the r.h.s. being clearly less than one. This rather paradoxical result stems from the assumption that altruism is expressed only through investment in education, parents making transfers of the aggregate good neither *inter vivos* nor as bequests.

6 Conclusions

It is not difficult to think of conditions that will keep a society in a state of permanent backwardness. Unremitting warfare and communicable diseases in the absence of public health measures, together with the privation that accompanies warfare and diseases, will surely suffice to bring about a Hobbesian existence, even when productive technologies are available. What we have established, however, is that there are stationary constellations of war losses and premature adult mortality such that both backwardness, in the sense there is no investment in human capital through schooling, and steady growth with a fully educated population are possible equilibria. The associated poverty trap is thereby precisely characterised.

Parents' altruism towards their children can exert a decisive influence on the outcome. If sufficiently strong, it can rule out backwardness in environments in which the hazards of destruction are such as to keep a selfish population in that condition for good. That is no great surprise. Where attaining – and maintaining – steady state growth is concerned, however, a rather paradoxical result holds. If, for whatever reason, parents express their altruism only in the form of investment in their children's education, their preferences over the resulting outcome must be more weakly concave than those over consumption. A strong measure of selfishness, which implies that provision for old age is the dominant or sole motive for investment, may then ensure the existence of a such a steady-state growth path when a fair measure of altruism would rule it out. Balanced growth must hold in all respects.

To close, it must be remarked that we have not explored environments in which cohort mortality rates and war losses are stochastic. To give a simple example, war may break out in the next period, and if it does so, some fraction of the capital stock will be destroyed and mortality will rise. The resulting stochastic dynamics will be the subject of future research.

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